Some New Results on Equal Sums of Like Powers

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Abstract. The Diophantine equation $\sum_{i=1}^{M} x_i^n = \sum_{i=1}^{M} y_i^n$ is examined for $n = 3$, 4 and 6 and $M = [(n+1)/2]$. A method for generating parametric solutions for $n = 4$ is derived and several new numerical examples for $n = 4, 6$ are given. The method also applies for all other values of M and possibly for values of n greater than 6, too.

1. In this article we describe a method to get many integral solutions of the type

$$\sum_{i=1}^{M} x_i^n = \sum_{i=1}^{M} y_i^n \tag{1}$$

from one known solution. While this method is general for any M, this article will be limited to cases where $M = [(n+1)/2]$.

For the case $n = 3$, that is, $M = 2$, the equation becomes:

$$x_1^3 + x_2^3 = y_1^3 + y_2^3 .$$

We solve the system of linear equations:

$$p_1 + q_1 = x_1 , \quad p_2 + q_1 = y_1 ,$$
$$p_2 + q_2 = x_2 , \quad p_1 - q_2 = y_2$$

for p_i and q_i. In general the system is characterized by the equations

$$p_i + q_i = x_i , \quad i = 1 \cdots M ,$$
$$p_{i+1} + q_i = y_i , \quad i = 1 \cdots M - 1$$

and

$$p_1 - q_M = y_M .$$

This last equation is included to make the determinant nonzero and thereby guarantee unique rational p_i's and q_i's from each numerical set of x_i's and y_i's.

Next we develop the equations:

$$\sum_{i=1}^{M} (p_i + \lambda q_i)^n - \sum_{i=1}^{M-1} (p_{i+1}\lambda + q_i)^n - (p_1 - \lambda q_M)^n = 0 . \tag{10}$$

We arrive at polynomials of the nth degree in λ. Because the p_i^n's always cancel and the q_i^n cancel whenever n is even, we are left with a polynomial of one degree lower for odd n and two degrees lower for even n in λ. We also know that the same polynomial has a solution $\lambda = 1$ which, when substituted gives us our initial numerical example:

$$\sum_{1}^{M} x_i^n = \sum_{1}^{M} y_i^n .$$

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Therefore we factor out $(\lambda - 1)$ and are left with a polynomial in λ which is two degrees lower than the original equation in the case of n odd and three degrees lower in the case of n even. If one of the roots of the remaining polynomial in λ is rational, it can then be used in Eq. (10) to generate a new numerical example.

For example, for the cases $n = 3$ and $n = 4$, this method is sufficient to give us another numerical example from any initial case because we are left with a linear equation in λ. Since we can interchange the x_i in an even polynomial with $-x_i$, and in an odd polynomial with $-y_i$, we obtain many more numerical examples from a given one, which might or might not coincide.

From the equation of the third order: $3^3 + 4^3 = -5^3 + 6^3$ we obtain twelve numerical examples:

$$
\begin{aligned}
&(-38, 87, 79, 48),\\
&(18, 19, 28, -21),\\
&(-177, 406, 343, 276),\\
&(-162, 229, 157, 192),\\
&(-65, 156, 142, 87),\\
&(15, -2, 16, -9)
\end{aligned}
$$

and the other six degenerate to the initial case.

In the case of $n = 4$, we take as our initial example

$$133^4 + 134^4 = 59^4 + 158^4$$

and obtain the following eight other numerical examples:

(11)	$12505169907^4 +$	$783453421^4 =$	$7038985479^4 +$	12178821457^4
(12)	$1^4 +$	$2^4 =$	$2^4 +$	1^4
(13) [1]	$111637^4 +$	$114613^4 =$	$34813^4 +$	134413^4
(14)	$3687711^4 +$	$6565526^4 =$	$1967986^4 +$	6710751^4
(15)	$1137493^4 +$	$654854^4 =$	$1167518^4 +$	60779^4
(16) [2]	$10381^4 +$	$10203^4 =$	$2903^4 +$	12231^4
(17)	$1453319^4 +$	$829418^4 =$	$461882^4 +$	1486969^4
(18) [3]	$1054067^4 +$	$545991^4 =$	$1057167^4 +$	522059^4 .

2. When x_i and y_i are functions of a parameter, i.e., in the case where we start with a general two parametric formula for the solution of Eq. (1) the method described in Section 1 can also be used to obtain additional general formulas for the solutions. For example [4]:

$$
\begin{aligned}
x_1 &= a^7 + a^5b^2 - 2a^3b^4 + 3a^2b^5 + ab^6\\
x_2 &= a^6b - 3a^5b^2 - 2a^4b^3 + a^2b^5 + b^7\\
y_1 &= a^7 + a^5b^2 - 2a^3b^4 - 3a^2b^5 + ab^6\\
y_2 &= a^6b + 3a^5b^2 - 2a^4b^3 + a^2b^5 + b^7 .
\end{aligned}
\tag{20}
$$

From this, if we now define p's and q's as in Section 1 and solve for the λ's in terms of a's and b's, we obtain the following four formulas:

$$f(a, b)_1 = a \tag{21}$$

$$\begin{aligned} f(a, b)_2 = {} & -a^{13} + a^{12}b + a^{11}b^2 + 5a^{10}b^3 + 6a^9b^4 - 12a^8b^5 - 4a^7b^6 \\ & + 7a^6b^7 - 3a^5b^8 - 3a^4b^9 + 4a^3b^{10} + 2a^2b^{11} - ab^{12} + b^{13} \end{aligned} \tag{22}$$

$$\begin{aligned} f(a, b)_3 = {} & a^{19} + 6a^{17}b^2 - 18a^{15}b^4 + 6a^{14}b^5 - 5a^{13}b^6 + 12a^{12}b^7 \\ & - 12a^{11}b^8 + 36a^{10}b^9 - 24a^9b^{10} - 12a^8b^{11} + 19a^7b^{12} \\ & + 36a^6b^{13} + 6a^5b^{14} + 12a^4b^{15} - 6a^3b^{16} + 6a^2b^{17} + ab^{18} \end{aligned} \tag{23}$$

$$\begin{aligned} f(a, b)_4 = {} & a^{31} - a^{30}b + 11a^{29}b^2 + a^{28}b^3 + 42a^{27}b^4 + 24a^{26}b^5 \\ & - 19a^{25}b^6 - 32a^{24}b^7 - 154a^{23}b^8 - 254a^{22}b^9 + 266a^{21}b^{10} \\ & + 718a^{20}b^{11} + 126a^{19}b^{12} - 303a^{18}b^{13} - 478a^{17}b^{14} \\ & - 830a^{16}b^{15} + 770a^{15}b^{16} + 916a^{14}b^{17} - 738a^{13}b^{18} \\ & + 21a^{12}b^{19} + 350a^{11}b^{20} - 434a^{10}b^{21} + 50a^9b^{22} + 142a^8b^{23} \\ & - 91a^7b^{24} + 76a^6b^{25} + 15a^5b^{26} - 3a^4b^{27} + 8a^3b^{28} \\ & - 8a^2b^{29} + ab^{30} - b^{31} \end{aligned} \tag{24}$$

where $x_1 = f(a, b)_n$, $x_2 = f(b, -a)_n$, $y_1 = f(a, -b)_n$ and $y_2 = f(b, a)_n$. For the numerical values $a = 2$, $b = 1$, Eqs. (21), (22), (23), and (24), give the numerical examples (12), (16), (17), and (11) respectively.

It is interesting to note that all these four formulas are of the power $6n + 1$. The other four numerical examples are given by the following formula:

$$\begin{aligned} x_1 = {} & a^{18}b + 3a^{17}b^2 - 15a^{16}b^3 + 15a^{15}b^4 + 6a^{14}b^5 - 45a^{13}b^6 + 82a^{12}b^7 \\ & - 15a^{11}b^8 - 123a^{10}b^9 + 171a^9b^{10} - 159a^8b^{11} + 159a^7b^{12} - 98a^6b^{13} \\ & + 30a^5b^{14} - 12a^4b^{15} + 3a^2b^{17} + b^{19} \\ x_2 = {} & a^{19} - a^{18}b - 3a^{17}b^2 - 3a^{16}b^3 + 21a^{15}b^4 - 12a^{14}b^5 - 44a^{13}b^6 \\ & + 86a^{12}b^7 - 93a^{11}b^8 + 87a^{10}b^9 + 3a^9b^{10} - 135a^8b^{11} + 142a^7b^{12} \\ & - 100a^6b^{13} + 72a^5b^{12} - 36a^4b^{15} + 12a^3b^{16} - 9a^2b^{17} + ab^{18} - b^{19} \\ y_1 = {} & a^{19} - a^{18}b - 3a^{17}b^2 - 3a^{16}b^3 + 21a^{15}b^4 - 6a^{14}b^5 - 44a^{13}b^6 \\ & + 62a^{12}b^7 + 15a^{11}b^8 - 129a^{10}b^9 + 165a^9b^{10} - 129a^8b^{11} + 88a^7b^{12} \\ & - 46a^6b^{13} + 18a^5b^{14} - 6a^4b^{15} + 12a^3b^{16} - 3a^2b^{17} + ab^{18} - b^{19} \\ y_2 = {} & a^{18}b - 3a^{17}b^2 + 3a^{16}b^3 + 21a^{15}b^4 - 60a^{14}b^5 + 27a^{13}b^6 + 58a^{12}b^7 \\ & - 75a^{11}b^8 + 57a^{10}b^9 - 63a^9b^{10} + 63a^8b^{11} - 87a^7b^{12} + 100a^6b^{13} \\ & - 66a^5b^{14} + 36a^4b^{15} - 18a^3b^{16} + 9a^2b^{17} + b^{19}. \end{aligned} \tag{25}$$

For the values $a = 2, b = 1$; $a = -2, b = 1$; $a = 1, b = 2$; $a = 1, b = -2$, Eq. (25) gives the numerical examples (13), (14), (15) and (18). The formulas (22) and (25) have been given by Lander [3] previously.

3. For the case $n = 6$, $m = 3$, the equation is of the third order and therefore has at least one real solution. This real solution need not be rational. Rational solutions to the λ equation are found by a trial factoring method.

By factoring the λ polynomial and taking the first known example:

$$23^6 + (\pm 10)^6 + (\pm 15)^6 = (\pm 3)^6 + (\pm 19)^6 + (\pm 22)^6 , \tag{31) [5}$$

we obtain eighteen new solutions, sixteen of which are trivial solutions of the form $a^6 + b^6 + c^6 = (\pm a)^6 + (\pm b)^6 + (\pm c)^6$ and their permutations.

The remaining two are:

$$81^6 + 50^6 + 37^6 = 65^6 + 78^6 + 11^6 \tag{32}$$

and

$$32^6 + 43^6 + 81^6 = 3^6 + 55^6 + 80^6 . \tag{33) [6}$$

Other solutions, which do not seem to have been previously recorded, obtained by the same method, are:

$$275^6 + 36^6 + 179^6 = 65^6 + 276^6 + 169^6 \tag{34}$$

$$211^6 + 125^6 + 300^6 = 68^6 + 289^6 + 249^6 \tag{35}$$

$$1^6 + 515^6 + 500^6 = 556^6 + 197^6 + 409^6 \tag{36}$$

$$148^6 + 249^6 + 103^6 = 188^6 + 243^6 + 11^6 \tag{37}$$

$$539^6 + 412^6 + 643^6 = 497^6 + 652^6 + 449^6 . \tag{38}$$

Attempts to find a parametric expression for $n > 6$ have thus far been fruitless.

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